JOURNAL OF MATERIALS SCIENCE 13 (1978) 1381–1387

The transverse compression of oriented nylon and polyethylene extrudates

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A theoretical treatment is given for the total diametral compression of a transversely isotropic elastic cylinder under compression between parallel rigid plates under conditions of plane strain. Experimental results are presented for the compression of isotropic and oriented cylinders of nylon 6.6 and linear polyethylene (Rigidex). These results confirm that the theoretical treatment is valid to a good degree of approximation. In the case of linear polythylene, where data are presented for ultra-highly oriented extrudates, the results are of some interest with regard to the elastic properties of these unusual materials.

1. Introduction

The compression of elastic spheres or cylinders was first considered by Hertz [1] and has subsequently been studied extensively, in view of its importance in technological problems involving roller bearings and gear wheels (see, for example, [2]). In general, discussion of the contact problem has been restricted in two ways. Firstly, only isotropic elastic spheres or cylinders have been considered. With the development of synthetic fibres and extruded oriented polymers there is however considerable interest in applying the compression problem more widely to anisotropic solids. The compression of a transversely isotropic elastic cylinder was shown to be a simple extension of the original Hertz contact problem and used to determine the transverse modulus of fibre monofilaments [3, 4]. Secondly, usually only the contact width and stresses close to the contact zone are calculated. In this paper, the Hertz contact problem will be extended to provide a solution for the total diametral compression of a transversely isotropic elastic cylinder under compression between parallel rigid plates. We have also undertaken measurements of the contact width and the total diametral compression for both isotropic and oriented rods to nylon 6.6 and linear polyethylene to confirm the validity of the theoretical treatment. For completeness, the measurements of contact width include data on linear polyethylene of very high extrusion ratios, where the samples were too small for accurate determination of the total compression. The results can be considered in terms of the determination of the transverse modulus for these materials, which is also of considerable interest.

2. Theory

2.1. Introduction

In considering the compression of a cylinder between two rigid parallel planes, it is customary to follow the Hertz method and assume that the width of the contact strip is small compared with the dimensions of the contacting bodies. It is then only necessary to consider the actual contact area between the cylinder and one of the planes, and treat the problem as for two semi-infinite solids in contact under conditions of plane strain. This treatment enables the contact width to be calculated in terms of the elastic constants, the applied load and the radius of the cylinder. It does, however, only consider the deformations in the contact zone, and no attempt is made to satisfy the boundary conditions on the surface of the cylinder.

In this paper the contraction of the cylinder along the diameter perpendicular to the planes of contact is considered. It is then necessary to consider the deformation outside the contact zone and to satisfy the boundary conditions on the surface of the cylinder. The situation under discussion is shown in Fig. 1.

2.2. Constitutive relations

For a general elastic solid the relationships between stresses σ_i and strains e_j are given by the generalized Hooke's law

$$\sigma_i = c_{ij} e_j, \tag{1}$$

$$e_j = s_{ji} \sigma_i, \tag{2}$$

where c_{ij} and s_{ji} are the stiffness and compliance constants respectively and i, j take the values 1, 2, 3 to 6. For an isotropic solid there are only two independent elastic constants and Equation 2 reduces to

$$e_j = \begin{pmatrix} s_{11} & s_{12} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{12} & s_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(s_{11} - s_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(s_{11} - s_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(s_{11} - s_{12}) \end{pmatrix} \sigma_i. \tag{3}$$

Expressing the stresses and strains in conventional Cartesian notation typical relationships take the form

$$e_{xx} = \frac{\partial u_x}{\partial x} = s_{11}\sigma_{xx} + s_{12}\sigma_{yy} + s_{12}\sigma_{zz}, \text{ etc.}$$

for the compliance constants, and

$$\sigma_{xx} = c_{11}\frac{\partial u_x}{\partial x} + c_{12}\frac{\partial u_y}{\partial y} + c_{12}\frac{\partial u_z}{\partial z}, \text{ etc.}$$

for the stiffness constants.

For an elastic solid showing transverse isotropy, the axis of transverse isotropy being the z-axis, there are five independent elastic constants and the equation corresponding to Equation 3 for an isotropic elastic solid takes the form

$$e_p = \begin{pmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{13} & 0 & 0 & 0 \\ s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(s_{11} - s_{12}) \end{pmatrix} \sigma_q \tag{4}$$

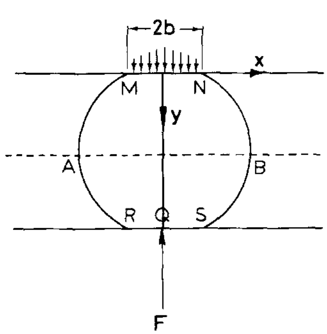

Figure 1 The compression problem.

Since we are considering a problem of plane strain, $u_z = 0$ and u_z, u_y are functions of x and y only.

In particular $\sigma_{yz} = \sigma_{zx} = 0$, and

$$\frac{\partial u_z}{\partial z} = 0. \tag{5}$$

For the isotropic cylinder Equation 5 gives

$$\sigma_{zz} = \frac{-s_{12}}{s_{11}}(\sigma_{xx} + \sigma_{yy}), \tag{6}$$

and for the transversely isotropic cylinder

$$\sigma_{zz} = \frac{-s_{13}}{s_{33}}(\sigma_{xx} + \sigma_{yy}). \tag{7}$$

It will be noted that Equations 6 and 7 are identical in form for the two cases.

2.3. Equations of equilibrium and compatibility

The equations of equilibrium are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0,$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0, \tag{8}$$

and

$$\frac{\partial \sigma_{zz}}{\partial z} = 0.$$

The compatibility conditions provide a further relationship involving σ_{xx} and σ_{yy}.

Rewriting Equation 8 in terms of the displacements, using the constitutive relations and the plane strain condition $u_z = 0$, $\partial u_z/\partial z = 0$, it may be shown that for this case of a two dimensional stress field, the transversely isotropic case is identical to the isotropic case and we have

$$\nabla^2 \left\{ \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right\} = 0.$$

Furthermore

$$\sigma_{xx} + \sigma_{yy} = (c_{11} + c_{12}) \left\{ \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right\},$$

and it follows that in both the isotropic and transversely isotropic case

$$\nabla^2 (\sigma_{xx} + \sigma_{yy}) = 0, \tag{9}$$

which is identical to the result for the compression of an isotropic cylinder.

2.4. Boundary conditions and stress functions

The results of Sections 2.2 and 2.3 show that the stress functions for σ_{xx}, σ_{yy} and σ_{xy} must satisfy Equations 8 and 9 and that σ_{zz} is given in terms of σ_{xx} and σ_{yy} by Equations 6 and 7 for the isotropic and transversely isotropic case, respectively. This suggests that the stress functions for the isotropic and transversely isotropic cases will take an identical form, any difference depending only on the difference in compliance coefficients for the two cases, which affects the numerical value of σ_{zz} through Equation 6 or 7.

The Hertz solution to the contact problem, assuming contact of two semi-infinite solids, requires that the stress functions satisfy the following conditions in the contact zone.

First it is assumed that to prevent interpenetration the normal displacement of the cylinder within the contact zone $-b < x < b$ is of the form $(u_y)_{y=0} = A + Bx$.

In addition the following stress conditions must be satisfied. In the contact zone when $y = 0$, the load per unit length of cylinder $F = -\int_{-b}^{b} \sigma_{yy}\, dx$ and $\sigma_{xy} = 0$. Outside the contact zone $\sigma_{yy} = 0$, when $y = 0$ and $|x| > b$. It is also assumed that the stress functions σ_{xx}, σ_{yy}, etc. vanish at infinity.

Hertz showed that these boundary conditions were satisfied by assuming that the stress distribution over the contact zone was satisfied by stress functions of an elliptical form.

A first approximation to the solution of the present problem would be to ignore satisfying the boundary conditions on the surface of the cylinder and accept the Hertz solution as valid throughout the cylinder. A better approximation is proposed along the following lines.

Consider the deformation of the upper half of the cylinder bounded by the diameter AB as shown in Fig. 1. Assume that there is a distributed load of the Hertzian form over the contact zone MN, but that the distributed load over the contact strip RS can be replaced by a concentrated load F at Q. The justification for the latter assumption is that the stresses given by the Hertz solution clearly reduce to this when $y \gg b$.

For a cylinder under compression by concentrated loads F at the two ends of a diameter, the condition of zero stress on the surface of the cylinder is met by applying an isotropic tension of magnitude $F/\pi R$ in the x–y plane. This solution would also hold for symmetrically arranged distributed loads of total magnitude F. It would therefore appear to be a reasonable approximation to the case under present consideration.

Along the diameter of the cylinder perpendicular to the plane of contact (i.e. the line $x = 0$) the stress functions obtained from the Hertz solution [2, 5–7] are

$$\sigma_{xx} = \frac{-2F}{\pi b^2}\left\{2\sqrt{(b^2+y^2)} - 2y - \frac{b^2}{\sqrt{(b^2+y^2)}}\right\},$$

$$\sigma_{yy} = \frac{-2F}{\pi}\frac{1}{\sqrt{(b^2+y^2)}}. \tag{10}$$

For a cylinder under compression by two concentrated loads of magnitude F the stresses along the line $x = 0$ [8] are

$$\sigma_{xx} = +\frac{F}{\pi R},$$

$$\sigma_{yy} = \frac{-2F}{\pi(2R-y)} - \frac{2F}{\pi y} + \frac{F}{\pi R}. \tag{11}$$

The proposed solution which assumes a distributed load over the contact zone MN and a concentrated load at Q takes the form

$$\sigma_{xx} = -\frac{2F}{\pi b^2}\left\{2\sqrt{(b^2+y^2)} - 2y - \frac{b^2}{\sqrt{(b^2+y^2)}}\right\} + \frac{F}{\pi R}$$

$$\sigma_{yy} = -\frac{2F}{\pi}\left\{\frac{1}{\sqrt{(b^2+y^2)}} + \frac{1}{(2R-y)}\right\} + \frac{F}{\pi R}. \tag{12}$$

2.5. Calculation of total compression

The total compression of the cylinder is given by

$$u = -2\int_0^R \frac{\partial u_y}{\partial y}\,dy.$$

We shall derive this quantity for the transversely isotropic cylinder; the isotropic case can then immediately be obtained by equating particular compliance constants. Thus for a transverely isotropic cylinder

$$\frac{\partial u_y}{\partial y} = \left(s_{12} - \frac{s_{13}^2}{s_{33}}\right)\sigma_{xx} + \left(s_{11} - \frac{s_{13}^2}{s_{33}}\right)\sigma_{yy},$$

where σ_{xx}, σ_{yy} are given by Equation 12.

Substitution and integration gives

$$u = \frac{4F}{-\pi b^2}\left\{\left(s_{12} - \frac{s_{13}^2}{s_{33}}\right)(b^2\sinh^{-1}(R/b) + R/R^2 + b^2 - R^2 - b^2\sinh^{-1}(R/b))\right\}$$
$$\frac{-4F}{\pi}\left\{\left(s_{11} - \frac{s_{13}^2}{s_{33}}\right)(\sinh^{-1}(R/b) + \log_e 2)\right\}$$
$$+\frac{2F}{\pi}\left\{\left(s_{12} - \frac{s_{13}^2}{s_{33}}\right) + \left(s_{11} - \frac{s_{13}^2}{s_{33}}\right)\right\}.$$

When $R \gg b$ this reduces to

$$u = \frac{-4F}{\pi}\left(s_{11} - \frac{s_{13}^2}{s_{33}}\right)(0.19 + \sinh^{-1}(R/b), \tag{13}$$

where we take $\log_e 2 = 0.69$.

For the isotropic case $s_{11} = s_{33} = 1/E$, $s_{13} = -\nu/E$

$$u = \frac{-4F}{\pi}\left(\frac{1-\nu^2}{E}\right)(0.19 + \sinh^{-1}(R/b)). \tag{14}$$

This result for the compression of an isotropic cylinder is very similar to that previously obtained by Föppl [9]. He gives;

$$u = -\frac{4F}{\pi}\left(\frac{1-\nu^2}{E}\right)\left(1/3 + \log_e\left(\frac{2R}{b}\right)\right).$$

For $R \gg b$, $\log_e(2R/b) = \sinh^{-1}(R/b)$ and

$$u = \frac{-4F}{\pi}\left(\frac{1-\nu^2}{E}\right)(1/3 + \sinh^{-1}(R/b)),$$

which is very close to Equation 14. The reason for the difference is due to the fact that Föppl assumed a parabolic distribution of stress in the contact zone, whereas we have followed Hertz and assumed an elliptical stress distribution.

2.6. Contact width

The Hertz solution to the contact problem [1] gives the following expression for the contact width.

$$b = \sqrt{\left(\frac{4FR}{\pi}\frac{(1-\nu^2)}{E}\right)}. \tag{15}$$

The extension of this to the case of a transverely isotropic cylinder [3] is

$$b = \sqrt{\left(\frac{4FR}{\pi}\left(s_{11} - \frac{s_{13}^2}{s_{33}}\right)\right)} \tag{16}$$

3. Experimental

3.1. Sample preparation

Isotropic cylinders of nylon 6.6 and linear polyethylene (Rigidex) were machined from extruded rods obtained from Nylonic Engineering Co. Ltd. and B.P. Chemicals International Ltd., respectively. Oriented samples (~ 2 cm length x ~ 0.85 cm diameter) of nylon 6.6 and linear polyethylene were machined from 25 mm extrudates produced by hydrostatic extrusion at 165° C (Abdul Jawad [10]) and 100° C (Gibson *et al.* [11] and Gibson [12]) respectively, using a Fielding Platt hydrostatic extrusion machine externally operated by an intensifier constructed in the Physics Department [12]. 2.52 mm diameter extrudates of linear polyethylene were also used for transverse modulus measurements. Before the measurements the samples of nylon 6.6 were kept under vacuum at 125° C for ~ 7 days and then under vacuum at room temperature for 3 days. The molecular weight characteristics of the linear polyethylene used are given in Table I.

TABLE I Sample characterization

Rigidex grade	$\bar{M}_w$	$\bar{M}_n$	$\bar{M}_w/\bar{M}_n$
50	10450	6180	16.90
25	98800	12950	7.63
140/60	67000	13350	5.02

3.2. Apparatus and measurements

For small diameter extrudates (i.e. 2.52 mm) of Rigidex, a compression apparatus was used similar to that described previously [3]. In this apparatus a compression rig was installed on the stage of a microscope. The test specimen was compressed between two blocks of glass and the contact zone was viewed in reflected light. The contact zone width $2b$ was measured along the contact zone at different points at each load in a time varying from 1 to 2 min. After each measurement the sample was left unloaded for about 10 times the loading time before applying a new load.

For large diameter samples a "dead loading compression creep apparatus" was used. This apparatus has been described in a recent publication [13]. The test specimen was compressed between two blocks of glass which were sufficiently thick for distortion during loading to be negligible, and were held in a compression cage between two sub-press plates. The loading procedure was followed in the same way as described previously [13]. The total diametral compression was measured by a transducer mounted on the upper press. The contact width $2b$ was measured simultaneously on the same sample as follows;

Before each experiment the surfaces of the specimen and the glass blocks were cleaned. The upper side of the specimen was coated with a narrow thin film of ink where the specimen was viewed in transmitted light. The contact width was measured by reflecting an image of the contact zone seen in transmitted light onto a graduated eye piece attached to a travelling microscope at a fixed distance from the compression cage. The contact width and total compression were measured simultaneously in a time of 10 to 20 sec. After each measurement the specimen was left unloaded for about 5 min before applying a new load. The width of the contact zone and the total compression at each load were reproducible within ± 10%. All the measurements were carried out at room temperature (17 ± 1° C).

4. Results

The experimental values of contact width and total diametral compression for isotropic nylon 6.6 and Rigidex have been compared with those obtained using Equation 15 (contact width) and Equation 14 (total compression), taking the Young's modulus for isotropic nylon 6.6 as 1.6

TABLE II Contact width $2b$ (cm) and total compression u (μm) for isotropic nylon 6.6 (specimen length = 1.89 cm, diameter = 0.85 cm)

F (N cm^{-1})	$2b$ (calculated)	$2b$ (measured)	u (calculated)	u (measured)
184	0.050	0.043 ± 0.004	57	52 ± 5
246	0.058	0.062 ± 0.005	73	66 ± 7
308	0.066	0.069 ± 0.007	88	79 ± 8
377	0.073	0.076 ± 0.007	104	92 ± 10
431	0.077	0.08 ± 0.009	110	100 ± 12

TABLE III Contact width $2b$ (cm) and total compression u (μm) for isotropic Rigidex 50 (specimen length = 1.99 cm; diameter = 1.03 cm)

F (N cm^{-1})	$2b$ (calculated)	$2b$ (measured)	u (calculated)	u (measured)
176	0.052	0.05 ± 0.006	52	45 ± 5
235	0.061	0.056 ± 0.006	67	58 ± 6
293	0.068	0.062 ± 0.007	81	70 ± 9
352	0.074	0.07 ± 0.007	95	84 ± 9
412	0.08	0.075 ± 0.008	108	97 ± 11

TABLE IV $s_{11} - s_{13}^2/s_{33}$ $(\text{GN m}^{-2})^{-1}$ for hydrostatically extruded nylon 6.6 and Rigidex 50

Polymer	Specimen dimension; length × diameter (cm)	R^*	$s_{11} - s_{13}^2/s_{33}$	
			u	$2b$
nylon 6.6	1.89 × 0.85	2	1.00 ± 0.12	0.86 ± 0.1
	1.89 × 0.85	2.7	0.89 ± 0.12	0.85 ± 0.1
	1.89 × 0.85	3.7	0.88 ± 0.12	0.87 ± 0.1
Rigidex 50	1.98 × 1.01	5.4	0.63 ± 0.08	0.69 ± 0.08
	1.98 × 0.78	9.75	0.62 ± 0.08	0.70 ± 0.08

*R = extrusion ratio = the ratio of the cross-sectional area of the original billet to the cross-sectional area of the extrudate.

GN m^{-2} and for Rigidex $1.63\,\text{GN m}^{-2}$. These values were obtained from measurements on the Instron machine at room temperature and refer to 10 sec response at 0.1% strain. Since no measurements were made to find the Poisson's ratio for isotropic nylon 6.6 and Rigidex it was assumed to be 0.35 in both cases. The value assumed for Poisson's ratio is not critical, and if this value is incorrect it will introduce only a very small error in the calculated values of u and $2b$.

The calculated and measured values of u and $2b$ are given in Tables II and III for isotropic nylon 6.6 and Rigidex respectively. For extruded samples of nylon 6.6 and Rigidex the quantity $s_{11} - s_{13}^2/s_{33}$ calculated from the contact width and total compression measurements is given in Table IV.

Equation 16 may be written as

$$b^2 = \frac{4FR}{\pi} s_{33} \left\{ \frac{s_{11}}{s_{33}} - \left(\frac{s_{13}}{s_{33}}\right)^2 \right\}. \quad (17)$$

Assuming $s_{11}/s_{33} \gg (s_{13}/s_{33})^2$ for highly extruded samples, the transverse modulus $1/s_{11}$ was calculated for small diameter extrudates (~ 2.5 mm) of hydrostatically extruded Rigidex. The results are shown in Fig. 2.

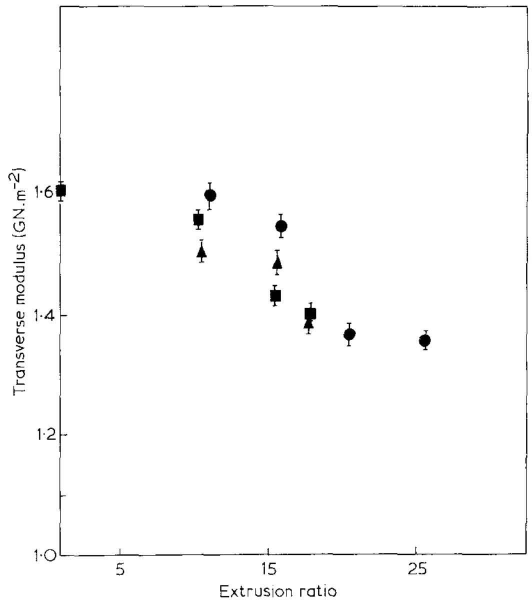

Figure 2 Transverse modulus as a function of extrusion ratio for hydrostatically extruded linear polyethylene. ● Rigidex 50, ▲ Rigidex 25, ■ 140/60 grade.

5. Discussion

It can be seen from Tables II and III that the comparison between experiment and theory for isotropic nylon 6.6 and Rigidex is good. Reproducibility of contact width and total compression was generally about ± 10%. Table IV shows that there is good agreement between the values of $s_{11} - s_{13}^2/s_{33}$ from contact width and total compression measurements. These results show that the proposed theoretical extension of the Hertz contact solution to a transversely isotropic cylinder is valid to a good degree of approximation.

In view of the interest in ultra-high modulus oriented polyethylene the results shown in Fig. 2 are worthy of comment. It appears that the transverse modulus of linear polyethylene decreases slightly with increasing extrusion ratio. This result is consistent with that found previously [3] for drawn linear polyethylene at lower draw ratios. It is of particular interest to note the small change in the transverse modulus compared with the change in the Young's modulus, which increases by a factor of twenty over the same range of draw ratios [11]. The final significance of this result will be discussed elsewhere, in the light of information from structural studies of these materials.

Acknowledgement

We are indebted to Professor K.L. Johnson, Department of Engineering, University of Cambridge, for his advice regarding the theoretical treatment presented for calculation of the total diametral compression.

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Received 8 June and accepted 14 October 1977.